

# On the maximum principle for parabolic equations with unbounded coefficients

*A.I. Nazarov\**,

St.Petersburg Dept of Steklov Institute and St.Petersburg University,

e-mail: al.il.nazarov@gmail.com

## 1 Introduction

This text is based on the paper [N87] and the note [N88] published in Russian in collected volumes by the Institute of Mathematics, Siberian Branch of USSR Academy of Sciences. Later it turned out that the proofs in [N87] can be essentially simplified. In particular, high-level arguments from [BL] and [KP] can be avoided (see in this connection [N01]). Also we fixed some gaps in proofs of auxiliary assertions.

We consider a priori maximum estimates for solution of initial-boundary value problem to parabolic equation

$$\mathcal{L}u := \sigma(x, t)D_t u - a_{ij}(x, t)D_i D_j u + b_i(x, t)D_i u + c(x, t)u = f(x, t) \quad (1.1)$$

in terms of the right-hand side in various spaces. Here and elsewhere we adopt the convention regarding summation with respect to repeated indices.

Such estimates for the Dirichlet problem to elliptic equations were established by A. D. Aleksandrov [Al], [Al1]. N. V. Krylov [Kr1], [Kr2] obtained these estimates for parabolic equations via  $\|f\|_{n+1, Q}$  provided all coefficients are bounded. N. N. Uraltseva and author [NU] succeeded to replace this assumption for  $b_i$  by  $b_i \in L_{n+1}(Q)$ . Similar results were independently obtained by Kai-sing Tso [Ts] using a different method. Finally, N. V. Krylov [Kr4] unified the estimates of [NU], [Ts]. Also he obtained the estimate via  $\|f\|_{n+1, Q}$  provided  $b_i \in L_n^x L_\infty^t(Q)$ , and similar estimates via  $\|f\|_{p+1, Q}$ ,  $p \geq n$ .

We establish the estimates of the same type in the space scale  $L_p^x L_q^t$  (for  $p \leq q$ ) or  $L_q^t L_p^x$  (for  $p \geq q$ ) with arbitrary  $p, q \leq \infty$  subject to  $\frac{n}{p} + \frac{1}{q} \leq 1$ . Coefficients  $b_i$  are assumed to belong to a space of the same type, maybe with different  $p$  and  $q$ . Moreover, we can manage the “composite” coefficients

$$b_i = \sum_{k=1}^m b_i^{(k)}, \quad b_i^{(k)} \in L_{p_k} L_{q_k}. \quad (1.2)$$

The paper is organized as follows. Section 2 is devoted to basic estimates. In Section 3 we prove the pivotal lemma and then derive the required estimates in non-degenerate case. In

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Section 4 we generalize these estimates for more wide class of operators. Also we prove the so-called Bony-type maximum principle. The estimate for operators with “composite” coefficients is proved in Section 5.

Let us recall some notation.  $x = (x_1, \dots, x_n)$  is a vector in  $\mathbb{R}^n$  with the Euclidean norm  $|x|$ ;  $(x; t)$  is a point in  $\mathbb{R}^{n+1}$ .

$B_R = \{x \mid |x| < R\}$  is a ball in  $\mathbb{R}^n$ .

$C_R = B_R \times \mathbb{R}^1$ ;  $C_{RT} = B_R \times ]0, T[$ .

$Q \subset C_{RT}$  is a domain in  $\mathbb{R}^{n+1}$ ;  $\Omega$  is the projection of  $Q$  to  $\mathbb{R}_x^n$ ;  $\overline{Q}$  is the closure of  $Q$ ;  $\chi_Q$  is the characteristic function of  $Q$ .

$|Q|$  and  $|\Omega|$  stand for the Lebesgue measure of corresponding dimension.

$\partial Q$  is boundary of  $Q$  while  $\partial' Q$  is its parabolic boundary that is the set of  $(x^0, t^0) \in \partial Q$  such that there exist  $\delta > 0$  and a function  $x(t) \in \mathcal{C}(\mathbb{R}^1)$  satisfying  $x(t^0) = x^0$  and  $(x(t), t) \in Q$  for  $t \in ]t^0, t^0 + \delta]$ . In particular, if  $Q = C_{RT}$  then  $\partial Q = B_R \times \{0\} \cup \partial B_R \times [0, T[$ .

By  $\sup_Q u$  we denote the essential supremum of a function  $u$  on a set  $Q$ . If  $u$  is continuous then  $Q_u = \{(x, t) \mid u > 0\}$ .

The symbol  $D_i$  denotes the operator of differentiation with respect to  $x_i$ ; in particular,  $Du = (D_1 u, \dots, D_n u)$  is the gradient of  $u$ .  $D_t u$  stands for the derivative of  $u$  with respect to  $t$ .

We always assume that in (1.1)  $\sigma \geq 0$ ,  $a_{ij} \lambda_i \lambda_j \geq 0$  for  $\lambda \in \mathbb{R}^n$ , and  $c \geq 0$ .  $\mathbf{Sp}(a)$  stands for the trace of the matrix  $a = (a_{ij})$ .

$\mathcal{C}(\overline{Q})$  is the space of continuous functions with the norm  $\|\cdot\|_Q$ .  $\mathcal{C}_0(\overline{Q})$  is the subspace of  $\mathcal{C}(\overline{Q})$  consisting of functions vanishing on  $\partial Q$ .  $\mathcal{C}^\infty(\overline{Q})$  is the set of smooth functions in  $\overline{Q}$ .

Let  $p, q \geq 1$  and let  $w(x, t) > 0$  a.e. in  $Q$ . We define  $L_p^x L_q^t[w](Q)$  as the space of (equivalence classes of) functions  $u$  such that the norm

$$\|u\| = \left[ \int_{\Omega} dx \left[ \int_0^T |wu|^q dt \right]^{\frac{p}{q}} \right]^{\frac{1}{p}}$$

is finite ( $u$  is assumed to be extended by zero on  $C_{RT} \setminus Q$ ). If  $p$  or  $q$  is infinite then corresponding integral should be replaced by sup. Analogously,  $L_q^t L_p^x[w](Q)$  is the space with norm in which integrals are taken in reverse order. If  $w \equiv 1$  it is omitted.

By Minkowski's inequality, for  $p < q$  the space  $L_p^x L_q^t(Q)$  is continuously embedded into  $L_q^t L_p^x(Q)$  (and  $L_p^t L_q^x(Q)$  is continuously embedded into  $L_q^x L_p^t(Q)$ ). For the sake of brevity we denote by  $\|\cdot\|_{p,q,(Q)}$  the norm in  $L_p^x L_q^t(Q)$  if  $p < q$ , and the norm in  $L_q^t L_p^x(Q)$  if  $p > q$ . Thus, it always stands for the stronger norm, the first index corresponds to the spatial variables and the second one – to the time variable. For  $p = q$  we evidently have  $L_p^t L_p^x(Q) = L_p^x L_p^t(Q) = L_p(Q)$ .

$W_{p,q}^{2,1}(Q)$  is the space with norm

$$\|u\|_{W_{p,q}^{2,1}(Q)} = \|u\|_{p,q,(Q)} + \|D_t u\|_{p,q,(Q)} + \|Du\|_{p,q,(Q)} + \|D(Du)\|_{p,q,(Q)}.$$

We set  $f_+ := \max\{f, 0\}$ ,  $f_- := \max\{-f, 0\}$  and denote by  $p'$  the Hölder conjugate exponent for  $p$ . We use letters  $M, N$  (with or without indices) to denote various constants. To indicate that, say,  $N$  depends on some parameters, we list them in the parentheses:  $N(\dots)$ .

## 2 Nondegenerate case. Basic estimates

In Sections 2 and 3 we suppose that

$$\delta \leq \sigma, c \leq \delta^{-1}; \quad |b| \leq \delta^{-1}; \quad \delta |\lambda|^2 \leq a_{ij} \lambda_i \lambda_j \leq \delta^{-1} |\lambda|^2, \quad \lambda \in \mathbb{R}^n$$

for some  $\delta > 0$ .

**Lemma 2.1.** *Let non-negative functions  $A, B \in W_{\infty}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  satisfy  $\mathcal{L}A \geq \mathbf{Sp}(a)$ ,  $\mathcal{L}B \geq |b|$  a.e. in  $Q$ . Then for all functions  $u \in W_{n+1}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq N_1(n)(B^2 + A)^{\frac{n}{2(n+1)}} \cdot \left\| \frac{(\mathcal{L}u)_+}{(\sigma \det(a))^{\frac{1}{n+1}}} \right\|_{n+1, (Q)}. \quad (2.1)$$

*Proof.* This statement is a particular case of [Kr4, Lemma 1.1].  $\square$

**Lemma 2.2.** *Under assumptions of Lemma 2.1, for all functions  $u \in W_{n,\infty}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq N_2(n)(B^2 + A)^{\frac{1}{2}} \cdot \left\| \frac{(\mathcal{L}u)_+}{(\det(a))^{\frac{1}{n}}} \right\|_{n,\infty, (Q)}. \quad (2.2)$$

*Proof.* We follow the scheme of proof of [Kr4, Lemma 3.3]. Let

$$f(x) = \chi_{\Omega} \cdot \sup_t \frac{(\mathcal{L}u)_+}{(\det(a))^{\frac{1}{n}}}.$$

We introduce a sequence  $f_k \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$  such that  $f_k \geq 0$ ,  $\|f_k - f\|_{n,(\mathbb{R}^n)} \rightarrow 0$  as  $k \rightarrow \infty$ .

By [Kr3, Theorem III.2.3], for arbitrary  $\beta > 0$  there exist  $\psi_k \in W_{\infty}^2(\mathbb{R}^n)$  such that

$$|D\psi_k(x)| \leq \psi_k(x)\beta^{\frac{1}{2}}; \quad 0 \leq \psi_k(x) \leq N_2(n)\beta^{-\frac{1}{2}}\|f_k\|_{n,(\mathbb{R}^n)}, \quad (2.3)$$

and for any non-negative matrix  $(\alpha_{ij})$

$$-\alpha_{ij}D_iD_j\psi_k + \beta\psi_k\mathbf{Sp}(\alpha) - f_k(\det(\alpha))^{\frac{1}{n}} \geq 0. \quad (2.4)$$

Now we consider functions

$$\xi_k = u - \psi_k - \|\psi_k\|_Q \cdot (\beta A + \beta^{\frac{1}{2}} B).$$

It is evident that  $\xi_k \in W_{n,\infty}^{2,1}(Q)$  and  $\xi_k|_{\partial'Q} \leq 0$ . We claim that  $\xi_k \leq N\|(\mathcal{L}\xi_k)_+\|_{n,\infty,(Q)}$  with  $N$  independent on  $\xi_k$ .

Indeed, let first  $\xi_k \in W_{\infty}^{2,1}(Q)$ . We introduce functions

$$\varphi_k = \delta^{-1}\chi_{\Omega} \cdot \sup_t (\mathcal{L}\xi_k)_+; \quad \tilde{\varphi}_k \in \mathcal{C}_0^{\infty}(B_{R+\delta^{-2}}); \quad \tilde{\varphi}_k > \varphi_k; \quad \|\tilde{\varphi}_k\|_{n,(\mathbb{R}^n)} \leq 2\|\varphi_k\|_{n,(\Omega)}.$$

Example VIII.2.2 in [Kr3] shows that there exists a solution  $v_k \leq 0$  of the boundary value problem for the Monge–Ampère equation

$$\det(D(Dv_k)) = \frac{1}{n^n} \tilde{\varphi}_k^n \quad \text{in } B_{R+\delta^{-2}}; \quad v_k|_{\partial B_{R+\delta^{-2}}} = 0.$$

Moreover,  $|v_k| \leq M\|\tilde{\varphi}_k\|_{n,(\mathbb{R}^n)}$  with  $M$  independent on  $\xi_k$ .

Since  $v_k$  is convex,  $|Dv_k(x)| \leq \delta^2|v_k(x)|$  in  $\Omega$ . This implies

$$-\mathcal{L}v_k \geq \mathbf{Sp}(a \cdot D(Dv_k)) - |b| \cdot |Dv_k| + c \cdot |v_k| \geq^* n \cdot (\det(a \cdot D(Dv_k)))^{\frac{1}{n}} \geq \delta \tilde{\varphi}_k \geq (\mathcal{L}\xi_k)_+$$

in  $Q$  (\* is the arithmetic-geometric means inequality).

Note that  $\xi_k + v_k \leq 0$  on  $\partial'Q$ . By the maximum principle (see, e.g., [Kr3, Lemma III.3.6]) we obtain  $\xi_k \leq |v_k| \leq 2\delta^{-1}M\|(\mathcal{L}\xi_k)_+\|_{n,\infty,(Q)}$ , and the claim follows. For  $\xi_k \in W_{n,\infty}^{2,1}(Q)$  we arrive at this estimate by approximation.

Inequalities (2.3) and (2.4) give

$$\mathcal{L}\xi_k \leq (\mathcal{L}u)_+ + a_{ij}D_iD_j\psi_k - b_iD_i\psi_k - \|\psi_k\|_Q \cdot (\beta\mathbf{Sp}(a) + \beta^{\frac{1}{2}}|b|) \leq (\mathcal{L}u)_+ - f_k(\det(a))^{\frac{1}{n}},$$

and therefore

$$\xi_k \leq N\|((\mathcal{L}u)_+ - f_k(\det(a))^{\frac{1}{n}})_+\|_{n,\infty,(Q)} \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

By (2.3) we have

$$u \leq (\beta A + \beta^{\frac{1}{2}}B + 1) \cdot N_2(n)\beta^{-\frac{1}{2}} \lim_k \|f_k\|_{n,(\mathbb{R}^n)}.$$

Finally, we minimize over  $\beta$ , and the Lemma follows.  $\square$

**Remark 2.1.** The norms in the right-hand side of (2.1) and (2.2) can be taken over the set  $Q_u$ . To prove it we can apply these estimates to  $Q_u$  instead of  $Q$ .

**Lemma 2.3.** *For all functions  $u \in W_{\infty}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq \left\| \frac{(\mathcal{L}u)_+}{c} \right\|_{\infty,(Q_u)}. \quad (2.5)$$

*Proof.* For  $w \equiv \left\| \frac{(\mathcal{L}u)_+}{c} \right\|_{\infty,(Q_u)}$  we have  $\mathcal{L}(u - w) \leq 0$  in  $Q_u$  and  $u \leq w$  on  $\partial'Q_u$ . By the maximum principle we get (2.5).  $\square$

**Lemma 2.4.** *For all functions  $u \in W_{\infty,1}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq \left\| \frac{(\mathcal{L}u)_+}{\sigma} \right\|_{\infty,1,(Q_u)}. \quad (2.6)$$

*Proof.* Denote by  $\Omega_u(\tau)$  the section of  $Q_u$  by the plane  $t = \tau$  and set

$$w(t) := \int_0^t \left\| \frac{(\mathcal{L}u(\cdot, \tau))_+}{\sigma(\cdot, \tau)} \right\|_{\infty,1,(\Omega_u(\tau))} d\tau.$$

Then  $\mathcal{L}(u - w) \leq 0$  in  $Q_u$  and  $u \leq w$  on  $\partial Q_u$ . By the maximum principle we get  $u \leq \max w$ , that gives (2.6).  $\square$

**Remark 2.2.** All estimates in Lemmata 2.1–2.4 have the form  $u \leq M\|(\mathcal{L}u)_+\|_{X(Q_u)}$ . If  $u|_{\partial'Q} = 0$  then we can apply these estimates also to  $-u$ . This gives four estimates of the form  $|u| \leq M\|\mathcal{L}u\|_{X(Q_u)}$ .

### 3 Nondegenerate case. Final estimates

We recall that we denote by  $\|\cdot\|_{p,q,(Q)}$  the norm in  $L_q^x L_p^t(Q)$  if  $p \geq q$  and the norm in  $L_p^x L_q^t(Q)$  if  $p \leq q$ . We also suppose that the assumptions from the beginning of Section 2 are fulfilled.

**Pivotal Lemma.** *Let  $\frac{n}{p} + \frac{1}{q} \leq 1$ , and let the functions  $A$  and  $B$  satisfy the assumptions of Lemma 2.1. Then for all  $u \in W_{p,q}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq N(n) \|B^2 + A\|_{Q_u}^{\frac{n}{2p}} \cdot \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q}} (\det(a))^{\frac{1}{p}} c^{1-\frac{n}{p}-\frac{1}{q}}} \right\|_{p,q,(Q_u)}. \quad (3.1)$$

*Proof.* We prove (3.1) in several steps.

**Step 1.** Suppose that  $Q = C_{RT}$  and  $a_{ij}$ ,  $b_i$ ,  $c$ ,  $\sigma$  are smooth. Then for smooth functions  $f$  the boundary value problem

$$\mathcal{L}u = f \quad \text{in } Q; \quad u|_{\partial'Q} = 0$$

is uniquely solvable, see, e.g., [F, Ch. 3]. Denote this solution by  $u = \mathcal{L}^{-1}f$ . Then  $\mathcal{L}^{-1}$  is evidently a linear operator from  $\mathcal{C}^\infty(\overline{Q})$  to  $\mathcal{C}_0(\overline{Q})$ .

**1a.** Let  $\frac{n}{p} + \frac{1}{q} = 1$  and  $p < q < \infty$ . Lemma 2.1 and Remark 2.2 show that  $\mathcal{L}^{-1}$  can be extended to the operator from  $L_{n+1}[(\sigma \det(a))^{-\frac{1}{n+1}}](Q)$  to  $\mathcal{C}_0(\overline{Q})$ , and

$$\|\mathcal{L}^{-1}\| \leq M_1 := N_1(n) \|B^2 + A\|_Q^{\frac{n}{2(n+1)}}.$$

Similarly, by Lemma 2.2 and Remark 2.2,  $\mathcal{L}^{-1}$  can be extended to the operator from the closure of  $\mathcal{C}^\infty(\overline{Q})$  in  $L_n^x L_\infty^t[(\det(a))^{-\frac{1}{n}}](Q)$  to  $\mathcal{C}_0(\overline{Q})$ , and<sup>1</sup>

$$\|\mathcal{L}^{-1}\| \leq M_2 := N_2(n) \|B^2 + A\|_Q^{\frac{1}{2}}.$$

Consider adjoint operator  $\mathcal{L}^{-1*}$  (with respect to duality  $\langle u, v \rangle = \int_Q uv$ ). It maps  $L_1(Q)$  (as a closed subspace of  $(\mathcal{C}_0(\overline{Q}))'$ ) to  $(L_{n+1}[(\sigma \det(a))^{-\frac{1}{n+1}}](Q))' = L_{\frac{n+1}{n}}[(\sigma \det(a))^{\frac{1}{n+1}}](Q)$ . Furthermore, it maps also  $L_1(Q)$  to  $L_{\frac{n}{n-1}}^x L_1^t[(\det(a))^{\frac{1}{n}}](Q) \subset (L_n^x \mathcal{C}^t[(\det(a))^{-\frac{1}{n}}](Q))'$ , since its image consists of functions. Its norms in these pairs do not exceed  $M_1$  and  $M_2$ , respectively.

By the Hölder inequality,

$$\|v \cdot \sigma^{\frac{1}{q}} (\det(a))^{\frac{1}{p}}\|_{L_{p'}^x L_{q'}^t(Q)} \leq \|v \cdot (\sigma \det(a))^{\frac{1}{n+1}}\|_{L_{\frac{n+1}{n}}(Q)}^\theta \cdot \|v \cdot (\det(a))^{\frac{1}{n}}\|_{L_{\frac{n}{n-1}}^x L_1^t(Q)}^{1-\theta},$$

where  $\theta = \frac{n+1}{q}$ . Therefore,  $\mathcal{L}^{-1*}$  maps  $L_1(Q)$  to  $L_{p'}^x L_{q'}^t[\sigma^{\frac{1}{q}} (\det(a))^{\frac{1}{p}}](Q)$ , and its norm does not exceed  $M_1^\theta M_2^{1-\theta}$ . This gives

$$\|\mathcal{L}^{-1}f\|_Q \leq N(n) \|B^2 + A\|_Q^{\frac{n}{2p}} \cdot \|\sigma^{-\frac{1}{q}} (\det(a))^{-\frac{1}{p}} f\|_{p,q,(Q)}, \quad (3.2)$$

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<sup>1</sup>Note that this closure coincides with the space  $L_n^x \mathcal{C}^t[(\det(a))^{-\frac{1}{n}}](Q)$ .

where  $N = \max\{N_1, N_2, 1\}$ .

**1b.** Let  $\frac{n}{p} + \frac{1}{q} = 1$  and  $q < p < \infty$ . By Lemma 2.4 and Remark 2.2,  $\mathcal{L}^{-1}$  can be extended to the operator from  $L_1^t \mathcal{C}^x[\sigma^{-1}](Q)$  to  $\mathcal{C}_0(\overline{Q})$ . Turning to adjoint operator and interpolating between  $L_{\frac{n+1}{n}}^t[(\sigma \det(a))^{\frac{1}{n+1}}](Q)$  and  $L_\infty^t L_1^x[\sigma](Q)$ , we again arrive at (3.2).

**1c.** Let  $\frac{n}{p} + \frac{1}{q} < 1$ , and  $p, q < \infty$ . We define  $p_0 = n + \frac{p}{q}$ ,  $q_0 = \frac{nq}{p} + 1$ , such that  $\frac{n}{p_0} + \frac{1}{q_0} = 1$  and  $\frac{p_0}{p} = \frac{q_0}{q}$ . By Lemma 2.3 and Remark 2.2,  $\mathcal{L}^{-1}$  can be extended to the operator from  $\mathcal{C}[c^{-1}](\overline{Q})$  to  $\mathcal{C}_0(\overline{Q})$ . From **1a** and **1b** one can see that it is continuous from the space

$$\begin{aligned} L_{p_0}^x L_{q_0}^t [\sigma^{-\frac{1}{q_0}} (\det(a))^{-\frac{1}{p_0}}](Q) & \text{ for } p \leq q; \\ L_{q_0}^t L_{p_0}^x [\sigma^{-\frac{1}{q_0}} (\det(a))^{-\frac{1}{p_0}}](Q) & \text{ for } p \geq q \end{aligned}$$

to  $\mathcal{C}_0(\overline{Q})$ . Turning to adjoint operator and interpolating, we arrive at

$$\|\mathcal{L}^{-1}f\|_Q \leq N(n)\|B^2 + A\|_{\overline{Q}}^{\frac{n}{2p}} \cdot \|\sigma^{-\frac{1}{q}} (\det(a))^{-\frac{1}{p}} c^{\frac{n}{p} + \frac{1}{q} - 1} f\|_{p,q,(Q)}. \quad (3.3)$$

**Step 2.** Let  $u$  be a smooth function,  $u|_{\partial'Q} \leq 0$ , and  $p, q < \infty$ .

**2a.** Suppose that  $Q$  and coefficients of operator are as in Step 1. We define  $u_1$  and  $u_2$  as solutions of boundary value problems

$$\begin{aligned} \mathcal{L}u_1 &= (\mathcal{L}u)_+ \quad \text{in } Q; \quad u_1|_{\partial'Q} = 0; \\ \mathcal{L}u_2 &= -(\mathcal{L}u)_- \quad \text{in } Q; \quad u_2|_{\partial'Q} = u|_{\partial'Q}. \end{aligned}$$

By the maximum principle  $u_2 \leq 0$ . Applying (3.2) or (3.3) to  $u_1$ , we obtain (3.1) with  $Q_u$  replaced by  $Q$ .

**2b.** Suppose that  $Q_u$  does not touch  $\partial' C_{RT}$ . We introduce a domain  $\tilde{Q}$  with piecewise smooth boundary such that  $Q_u \subset \tilde{Q} \subset C_{RT}$ . Then we consider a sequence of Lipschitz functions  $\zeta_k$  such that  $\zeta_k = (\mathcal{L}u)_+$  in  $\tilde{Q}$  and  $\zeta_k \downarrow (\mathcal{L}u)_+ \cdot \chi_{\tilde{Q}}$ .

Denote by  $u_k$  the solution of boundary value problem

$$\mathcal{L}u_k = \zeta_k \quad \text{in } C_{RT}; \quad u_k|_{\partial' C_{RT}} = 0.$$

Then evidently  $u_k \geq 0 \geq u$  on  $\partial'\tilde{Q}$ , and  $\mathcal{L}u_k \geq \mathcal{L}u$  in  $\tilde{Q}$ . By the maximum principle  $u_k \geq u$  in  $\tilde{Q}$ . We apply to  $u_k$  in  $C_{RT}$  the estimate obtained in **2a** and pass to the limit as  $k \rightarrow \infty$ . It gives us (3.1) with  $Q^u$  replaced by  $\tilde{Q}$ .

**2c.** Since  $p, q < \infty$ , we can extend this estimate to arbitrary admissible coefficients and functions  $u$  by approximation.

**2d.** For arbitrary  $Q_u$  we can consider functions  $u_\varepsilon = u - \varepsilon$  and approximate  $Q^u$  by domains  $Q_{u_\varepsilon} \subset \tilde{Q}_k \subset Q^u$  described in **2b**. Then we apply to  $u_\varepsilon$  in  $\tilde{Q}_k$  the estimate obtained in **2c**. Passage to the limit as  $k \rightarrow \infty$  and then as  $\varepsilon \rightarrow 0$  gives (3.1) in required form. The statement for  $p, q < \infty$  is proved.

**3.** The cases  $p = n$ ,  $q = 1$  and  $p = q = \infty$  are considered in Lemmata 2.2, 2.4 and 2.3, respectively.

**3a.** Let  $p = \infty$ ,  $1 < q < \infty$ . Then we consider the estimate (3.1) for  $\max\{q, nq'\} \leq p < \infty$ . Since  $\|\varphi\|_{p,q,(Q_u)} \leq \|\varphi\|_{\infty,q,(Q_u)} \cdot |\Omega|^{\frac{1}{p}}$ , we obtain

$$u \leq N(n) \|B^2 + A\|_{Q_u}^{\frac{n}{2p}} \cdot \delta^{-\frac{2n}{p}} \cdot |\Omega|^{\frac{1}{p}} \cdot \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q}} c^{1-\frac{1}{q}}} \right\|_{\infty,q,(Q_u)}.$$

Passage to the limit as  $p \rightarrow \infty$  gives (3.1) for  $p = \infty$ .

**3b.** In a similar way, if  $q = \infty$ ,  $n < p < \infty$ , then we consider the estimate (3.1) for large finite  $q$  and pass to the limit using the embedding  $\|\varphi\|_{p,q,(Q_u)} \leq \|\varphi\|_{p,\infty,(Q_u)} \cdot T^{\frac{1}{q}}$ .  $\square$

**Remark 3.1.** For  $p = q$  the estimate (3.1) was obtained by N.V. Krylov [Kr4] in direct way. Interpolation method clarifies the nature of the weight  $c^{\frac{n-p}{p+1}}$  in [Kr4]. The result of [Kr4, Lemma 3.3] for elliptic operators also can be obtained by interpolation between border spaces  $L_n[(\det(a))^{-\frac{1}{n}}]$  and  $\mathcal{C}[c^{-1}]$ .

**Corollary 3.1.** If there exists a function  $B$  satisfying assumptions of Lemma 2.1 then the assertion of Pivotal Lemma holds with  $\|B^2 + A\|_{Q_u}$  replaced by  $(\|B\|_{Q_u} + R)^2$ . This follows from Lemma 1.2 in [Kr4].

**Theorem 3.1.** *Let the assumptions in the beginning of Section 2 are satisfied. Suppose that  $\frac{n}{p_0} + \frac{1}{q_0} \leq 1$  and  $\frac{n}{p_1} + \frac{1}{q_1} \leq 1$ . We put*

$$h = \sigma^{-\frac{1}{q_1}} (\det(a))^{-\frac{1}{p_1}} c^{\frac{n}{p_1} + \frac{1}{q_1} - 1} |b|. \quad (3.4)$$

*Then for all functions  $u \in W_{p_0,q_0}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial'Q} \leq 0$  the following estimate holds:*

$$u \leq M^{\frac{n}{p_0}} \cdot \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q_0}} (\det(a))^{\frac{1}{p_0}} c^{1-\frac{n}{p_0}-\frac{1}{q_0}}} \right\|_{p_0,q_0,(Q_u)}, \quad (3.5)$$

where  $M$  depends only on  $n$ ,  $R$ ,  $p_1$  and the norm  $\|h\|_{p_1,q_1,(Q_u)}$ .

*Proof. 1.* Let  $p_0, q_0, p_1, q_1 < \infty$ . Then it is sufficient to obtain the estimate (3.5) for smooth coefficients and functions  $u$  and then to pass to the limit. Moreover, we can assume that  $Q_u$  does not touch  $\partial' C_{RT}$ .

As in the proof of Pivotal Lemma, Step **2b**, we approximate  $Q_u$  by a domain  $\tilde{Q}$  with piecewise smooth boundary such that  $Q_u \subset \tilde{Q} \subset C_{RT}$ . Then we introduce a sequence of operators  $\mathcal{L}_k$  with smooth coefficients, satisfying the assumptions of Theorem, such that  $\mathcal{L}_k = \mathcal{L}$  in  $\tilde{Q}$  and  $|b^{(k)}| \downarrow |b| \cdot \chi_{\tilde{Q}}$ .

Denote by  $B_k$  the solution of boundary value problem

$$\mathcal{L}_k B_k = |b^{(k)}| \quad \text{in } C_{RT}; \quad B_k|_{\partial' C_{RT}} = 0.$$

This function satisfies the assumptions of Lemma 2.1 for the operator  $\mathcal{L}_k$ . Therefore, we can apply the estimate (3.1) with  $p = p_1$ ,  $q = q_1$ , with regard to Corollary 3.1, to  $u = \pm B_k$ . This gives

$$\|B_k\|_{C_{RT}} \leq N(n) \cdot (\|B_k\|_{C_{RT}} + R)^{\frac{n}{p_1}} \|h_k\|_{p_1,q_1,(C_{RT})}$$

(here  $h_k$  is defined by (3.4) with  $b$  replaced by  $b_k$ ).

Note that  $q_1 < \infty$  implies  $p_1 > n$ . Therefore, if  $\|B_k\|_{C_{RT}} > R$  then

$$\|B_k\|_{C_{RT}} \leq \left[ 2^n N(n)^{p_1} \|h_k\|_{p_1, q_1, (C_{RT})}^{p_1} \right]^{\frac{1}{p_1 - n}}. \quad (3.6)$$

We substitute this estimate to (3.1), pass to the limit as  $k \rightarrow \infty$  and obtain the inequality (3.5) with  $M = N(n) \left( 2R + [2^n N(n)^{p_1} \|h\|_{p_1, q_1, (\tilde{Q})}^{p_1}]^{\frac{1}{p_1 - n}} \right)$ . Then we finish the proof as in Step **2d** of the proof of Pivotal Lemma.

**2.** The estimate (3.5) for  $p_0 = \infty$  evidently follows from Lemma 2.3 ( $q_0 = \infty$ ), Lemma 2.4 ( $q_0 = 1$ ) and Step **3a** in Pivotal Lemma ( $1 < q_0 < \infty$ ).

**3.** Let  $q_0 = \infty$ ,  $n < p_0 < \infty$ , and/or  $q_1 = \infty$ ,  $n < p_1 < \infty$ . Then, as in Step **3b** in Pivotal Lemma, we can consider the estimate (3.5) for large finite  $q_0$  ( $q_1$ ), use the embedding theorem and pass to the limit as  $q_0 \rightarrow \infty$  ( $q_1 \rightarrow \infty$ ).

**4.** Let  $1 < q_1 < \infty$ ,  $p_1 = \infty$ ,  $p_0 > n$ . Using the estimate (3.5) for large finite  $p_1$  and the embedding theorem, we arrive at

$$\begin{aligned} u &\leq N(n) \left( 2R + \left[ 2^n N(n)^{p_1} \left\| \frac{\sigma^{-\frac{1}{q_1}} c^{\frac{1}{q_1} - 1} |b|}{(\det(a))^{\frac{1}{p_1}} c^{-\frac{n}{p_1}}} \right\|_{\infty, q_1, (Q_u)}^{p_1} |\Omega|^{\frac{1}{p_1 - n}} \right]^{\frac{n}{p_0}} \right. \\ &\quad \times \left. \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q_0}} (\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)} \right). \end{aligned}$$

The expression in large brackets does not exceed

$$2R + 2^{\frac{n}{p_1 - n}} N(n)^{1 + \frac{n}{p_1 - n}} \delta^{-\frac{2n}{p_1 - n}} \left\| \sigma^{-\frac{1}{q_1}} c^{\frac{1}{q_1} - 1} |b| \right\|_{\infty, q_1, (Q_u)}^{1 + \frac{n}{p_1 - n}} |\Omega|^{\frac{1}{p_1 - n}}.$$

We push  $p_1 \rightarrow \infty$  and obtain

$$u \leq N(n) \left( 2R + N(n) \left\| \sigma^{-\frac{1}{q_1}} c^{\frac{1}{q_1} - 1} |b| \right\|_{\infty, q_1, (Q_u)} \right)^{\frac{n}{p_0}} \cdot \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q_0}} (\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)}. \quad (3.7)$$

Then, as in part **3**, we derive the desired estimate for  $p_1 = q_1 = \infty$ ,  $p_0 > n$ .

**5.** Now let  $q_1 = 1$ ,  $p_0 > n$ . Since for  $q > 1$  and  $\varphi \in L_\infty(Q_u)$  we have

$$\|\varphi\|_{\infty, q, (Q_u)} \leq \|\varphi\|_{\infty, (Q_u)}^{\frac{q-1}{q}} \cdot \|\varphi\|_{\infty, 1, (Q_u)}^{\frac{1}{q}},$$

the expression in brackets in (3.7) does not exceed

$$2R + N(n) \delta^{-4\frac{q_1-1}{q_1}} \left\| \sigma^{-1} |b| \right\|_{\infty, 1, (Q_u)}^{\frac{1}{q_1}}.$$

We push  $q_1 \rightarrow 1$  and obtain (3.7) for  $q_1 = 1$ .

In a similar way we consider the case  $p_0 = n$ ,  $p_1 > n$ . For  $u \in W_\infty^{2,1}(Q_u)$  we have from part **3**

$$u \leq M^{\frac{n}{p_0}} \cdot \left\| \frac{(\mathcal{L}u)_+}{(\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0}}} \right\|_{n, \infty, (Q_u)}^{\frac{n}{p_0}} \cdot \left\| \frac{(\mathcal{L}u)_+}{(\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0}}} \right\|_{\infty, (Q_u)}^{1 - \frac{n}{p_0}}.$$



Passage to the limit as  $p_0 \rightarrow n$  gives the desired estimate, and it remains to recall that  $W_\infty^{2,1}(Q_u)$  is dense in  $W_{n,\infty}^{2,1}(Q_u)$ .

**6.** The case  $p_1 = n$  is a special one since the inequality (3.6) fails. We construct a function  $B$  from Pivotal Lemma in a different way, see [Kr4, Section 3]. We introduce a function

$$f \in C_0^\infty(B_{R+\varepsilon}); \quad f(x) > \sup_t (h(x, t) \cdot \chi_{\Omega_u(t)}); \quad \|f\|_{n,(\mathbb{R}^n)} \leq 2\|h\|_{n,\infty,(Q_u)}.$$

Set  $B := -v$  where  $v$  is the solution of boundary value problem

$$\det(D(Dv)) = \left(\frac{2}{n}\right)^n f^n (1 + |Dv|^2)^{\frac{n}{2}} \quad \text{in } B_{R+\varepsilon}; \quad v|_{\partial B_{R+\varepsilon}} = 0.$$

Lemmata 3.1 and 3.2 in [Kr4] and Remark 3.1 in [Kr4] show that  $B$  satisfies the assumptions of Lemma 2.1, and

$$\|B\|_{C_{RT}} \leq N_3(n)(R + \varepsilon) \exp(N_4(n)\|f\|_{n,(\mathbb{R}^n)}^n).$$

Finally, we can push  $\varepsilon \rightarrow 0$ . □

**Remark 3.2.** As it is pointed in Introduction, Theorem 3.1 and more general Theorem 4.1 were proved by various methods for  $p_0 = q_0 = n + 1$ ,  $p_1 = q_1 = \infty$  (see [Kr2]); for  $p_0 = q_0 = p_1 = q_1 = n + 1$  (see [NU]); for  $p_0 = q_0$ ,  $p_1 = q_1$  or  $p_1 = n$  (see [Kr4]). See also [Al], [Al1] for the case  $p_0 = p_1 = n$ .

## 4 Generalization of Theorem 3.1

In this Section we weaken requirements for coefficients of the operator  $\mathcal{L}$  comparing to Sections 2 and 3.

**Theorem 4.1.** *Let  $\frac{n}{p_0} + \frac{1}{q_0} \leq 1$  and  $\frac{n}{p_1} + \frac{1}{q_1} \leq 1$ . Suppose that the following assumption (depending on  $p_0$  and  $q_0$ ) is satisfied a.e. in  $Q$ :*

$$\begin{aligned} \mathbf{Sp}(a) &> 0 && \text{if } p_0 = n; \\ \sigma &> 0 && \text{if } q_0 = 1; \\ c &> 0 && \text{if } p_0 = q_0 = \infty; \\ c + \sigma &> 0 && \text{if } p_0 = \infty, \quad 1 < q_0 < \infty; \\ \mathbf{Sp}(a) + c &> 0 && \text{if } q_0 = \infty, \quad n < q_0 < \infty; \\ \mathbf{Sp}(a) + \sigma &> 0 && \text{if } \frac{n}{p_0} + \frac{1}{q_0} = 1, \quad p_0, q_0 < \infty; \\ \mathbf{Sp}(a) + \sigma + c &> 0 && \text{if otherwise.} \end{aligned} \tag{4.1}$$

Let also  $\|h\|_{p_1, q_1, (Q)} < \infty$ , where the function  $h$  is defined in (3.4). Then for all functions  $u \in W_{p_0, q_0}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial' Q} \leq 0$ , the estimate (3.5) holds. The quantity  $M$  in (3.5) depends only on  $n$ ,  $R$ ,  $p_1$  and the norm  $\|h\|_{p_1, q_1, (Q_u)}$ , and we set  $0^0 = 1$ ,  $\frac{0}{0} = 0$ , if such expression arises.

*Proof. 1.* Let  $p_0, q_0 < \infty$  and  $\frac{n}{p_0} + \frac{1}{q_0} < 1$ . We set

$$\mathcal{L}_s u := \chi_{h \leq s} \cdot \mathcal{L} u + \chi_{h > s} \cdot (D_t u - \Delta u + u).$$

Let  $a_{ijs}, b_{is}, c_s, \sigma_s$  be the coefficients of  $\mathcal{L}_s$ . Then  $\mathcal{L}_s$  evidently satisfies assumptions of Theorem 4.1 with  $h_s = h \cdot \chi_{h \leq s}$ , and

$$\begin{aligned} & \left\| \frac{(\mathcal{L}_s u)_+}{\sigma_s^{\frac{1}{q_0}} (\det(a_s))^{\frac{1}{p_0}} c_s^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)} \\ & \leq \left\| \frac{(\mathcal{L} u)_+ \cdot \chi_{h \leq s}}{\sigma^{\frac{1}{q_0}} (\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)} + \left\| (D_t u - \Delta u + u)_+ \cdot \chi_{h > s} \right\|_{p_0, q_0, (Q_u)}. \end{aligned} \quad (4.2)$$

Since  $p_0, q_0 < \infty$ , the right-hand side of (4.2) tends to the norm in the right-hand side of (3.5) as  $s \rightarrow \infty$ . Thus, in this case it is sufficient to prove Theorem for  $h$  bounded.

It is evident that (3.5) does not change if we multiply all coefficients of  $\mathcal{L}$  by the same function positive almost everywhere. Thus, by (4.1) we can assume without loss of generality that  $\mathbf{Sp}(a) + \sigma + c = 1$  a.e. in  $Q$  and therefore all coefficients of  $\mathcal{L}$  are bounded.

For  $\varepsilon > 0$  we set

$$\mathcal{L}_\varepsilon u := \mathcal{L} u + \varepsilon \cdot (D_t u - \Delta u + u).$$

The operator  $\mathcal{L}_\varepsilon$  satisfies all assumptions of Theorem 3.1, and  $h_\varepsilon \leq h$ . Therefore, the estimate (3.5) holds for  $\mathcal{L}_\varepsilon$  instead of  $\mathcal{L}$ .

It remains to push  $\varepsilon$  to zero and to note that  $\mathbf{Sp}(a) + \sigma + c = 1$  a.e. in  $Q$  implies

$$\left\| \frac{\varepsilon \cdot (D_t u - \Delta u + u)_+}{(\sigma + \varepsilon)^{\frac{1}{q_0}} (\det(a + \varepsilon I))^{\frac{1}{p_0}} (c + \varepsilon)^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)} \leq M(u) \varepsilon^\delta \xrightarrow{\varepsilon \rightarrow 0} 0$$

(here  $\delta = \min\{\frac{1}{q_0}, \frac{1}{p_0}, 1 - \frac{n}{p_0} - \frac{1}{q_0}\}$  and  $I$  stands for identity matrix).

**2.** In the case  $\frac{n}{p_0} + \frac{1}{q_0} = 1$ ,  $p_0, q_0 < \infty$ , repeating the first step of the part **1**, we reduce the proof to the case of bounded  $h$  and  $\mathbf{Sp}(a) + \sigma = 1$  a.e. in  $Q$ .

For  $s > 0$ ,  $\varepsilon > 0$  we set  $c_s = \min\{c, s\}$ ;  $b_{is} = b_i \left(\frac{c_s}{c}\right)^{1 - \frac{n}{p_1} - \frac{1}{q_1}}$ ;

$$\mathcal{L}_{s\varepsilon} u := (\sigma + \varepsilon) D_t u - a_{ij}(x, t) D_i D_j u - \varepsilon \Delta u + b_{is} D_i u + (c + \varepsilon) u.$$

The operator  $\mathcal{L}_{s\varepsilon}$  satisfies all assumptions of Theorem 3.1, and the estimate (3.5) holds for  $\mathcal{L}_{s\varepsilon}$  instead of  $\mathcal{L}$ .

Since  $p_0, q_0 < \infty$ , we can pass to the limit as  $s \rightarrow \infty$ . Then, similarly to part **1**, using  $\mathbf{Sp}(a) + \sigma = 1$  we push  $\varepsilon$  to 0.

**3.** For  $p_0 = \infty$ ,  $1 < q_0 < \infty$  we can assume that  $\sigma + c = 1$  a.e. in  $Q$ . We apply the result of part **1** to the operator  $\mathcal{L}_\varepsilon$  for large finite  $p$ . By embedding  $L_{q_0}^t L_\infty^x(Q) \rightarrow L_{q_0}^t L_p^x(Q)$  we have

$$u \leq M^{\frac{n}{p}} \varepsilon^{-\frac{n}{p}} |\Omega|^{\frac{1}{p}} \cdot \left\| \frac{(\mathcal{L}_\varepsilon u)_+}{(\sigma + \varepsilon)^{\frac{1}{q_0}} (c + \varepsilon)^{1 - \frac{1}{q_0}}} \right\|_{\infty, q_0, (Q_u)}.$$

We pass to the limit as  $p \rightarrow \infty$ . Then, similarly to part **1**, using  $\sigma + c = 1$  we push  $\varepsilon$  to 0.

In a similar way, for  $q_0 = \infty$ ,  $n < p_0 < \infty$  we can assume that  $\mathbf{Sp}(a) + c = 1$  a.e. in  $Q$ . We apply the result of part 1 to  $\mathcal{L}_\varepsilon$  for large finite  $q$  and obtain

$$u \leq M^{\frac{n}{p_0}} \varepsilon^{-\frac{1}{q}} T^{\frac{1}{q}} \cdot \left\| \frac{(\mathcal{L}_\varepsilon u)_+}{(\det(a + \varepsilon I))^{\frac{1}{p_0}} (c + \varepsilon)^{1 - \frac{n}{p_0}}} \right\|_{p_0, \infty, (Q_u)}.$$

We pass to the limit as  $q \rightarrow \infty$  and then as  $\varepsilon \rightarrow 0$ .

In the same way, using these results we obtain the estimate for the case  $p_0 = q_0 = \infty$ .

4. Now let  $p_0 = n$ . Then we can assume that  $\mathbf{Sp}(a) = 1$  a.e. in  $Q$ . For  $p > n$ ,  $u \in W_{\infty}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  we apply the result of part 1 to  $\mathcal{L}_\varepsilon$  and arrive at

$$u \leq M^{\frac{n}{p}} \cdot \left\| \frac{(\mathcal{L}_\varepsilon u)_+}{(\det(a + \varepsilon I))^{\frac{1}{p}} (c + \varepsilon)^{1 - \frac{n}{p}}} \right\|_{\infty, (Q_u)}^{1 - \frac{n}{p}} \cdot \left\| \frac{(\mathcal{L}_\varepsilon u)_+}{(\det(a + \varepsilon I))^{\frac{1}{p}} (c + \varepsilon)^{1 - \frac{n}{p}}} \right\|_{n, \infty, (Q_u)}^{\frac{n}{p}}.$$

Passing to the limit as  $p \rightarrow n$  and then as  $\varepsilon \rightarrow 0$ , we obtain the desired statement in this case, since  $W_{\infty}^{2,1}(Q)$  is dense in  $W_{n, \infty}^{2,1}(Q)$ .

The case  $q_0 = 1$  is managed in a similar way.  $\square$

**Remark 4.1.** For  $p_0 = \infty$  the constant in (3.5) does not depend on  $h$ . However, a simple example shows that we cannot drop the restriction on  $h$ . Let

$$Q = C_{1,1}, \quad \mathcal{L}u = D_t u - \Delta u + \frac{(n+1)x_i}{|x|^\alpha} D_i u + u.$$

For  $\alpha < 2$  the operator  $\mathcal{L}$  satisfies the assumptions of Theorem 4.1 since  $\|h\|_{n, \infty, (Q)} < \infty$ . However, if  $\alpha = 2$  then the function  $U = 2t - t^2 - |x|^2 - \frac{1}{2}$  satisfies  $\mathcal{L}U < 0$  while  $U|_{\partial' Q} \leq 0$ ,  $U(0, 1) = \frac{1}{2}$ .

**Remark 4.2.** The assumption  $W_{p_0, q_0}^{2,1}(Q)$  in Theorem 4.1 can be replaced by  $W_{p_0, q_0, loc}^{2,1}(Q)$ . This fact can be proved as Lemma III.3.8 in [Kr3].

Now we weaken the assumption  $c \geq 0$ . For the sake of brevity we formulate only the simplest generalization.

**Theorem 4.2.** *Suppose that there is a constant  $\varkappa > 0$  such that  $c_\varkappa := c + \varkappa\sigma \geq 0$  and the assumptions of Theorem 4.1 are satisfied with  $c_\varkappa$  instead of  $c$ . Then for all functions  $u \in W_{p_0, q_0}^{2,1}(Q) \cap \mathcal{C}(\overline{Q})$  such that  $u|_{\partial' Q} \leq 0$  the following estimate holds:*

$$u \leq \exp(\varkappa T) \cdot M^{\frac{n}{p_0}} \cdot \left\| \frac{(\mathcal{L}u)_+}{\sigma^{\frac{1}{q_0}} (\det(a))^{\frac{1}{p_0}} c^{1 - \frac{n}{p_0} - \frac{1}{q_0}}} \right\|_{p_0, q_0, (Q_u)}.$$

*Proof.* Consider the function  $v = \exp(-\varkappa t)u$ . We have  $\mathcal{L}_\varkappa v := \mathcal{L}v + \varkappa\sigma v = \exp(-\varkappa t)\mathcal{L}u$ . We apply Theorem 4.1 to the operator  $\mathcal{L}_\varkappa$  and to the function  $v$ . Then we take into account inequalities  $\exp(-\varkappa t) \leq 1$  and  $\exp(\varkappa t) \leq \exp(\varkappa T)$ , and the statement follows.  $\square$

Finally we prove the Bony-type maximum principle. In the case of bounded coefficients it was proved in [Bo] for elliptic operators and in [Ts], [Kr4] for parabolic operators.

**Theorem 4.3.** *Let the assumptions of Theorem 4.1 be satisfied with  $p_0 = p_1$ ,  $q_0 = q_1$ . Suppose that a function  $u \in W_{p_0, q_0, \text{loc}}^{2,1}(Q)$  attains its non-negative maximum in an interior point of  $Q$ . Then*

$$\sup_Q \frac{\mathcal{L}u}{\mathbf{Sp}(a) + \sigma + c} \geq 0.$$

*Proof.* Without loss of generality we can assume that  $\mathbf{Sp}(a) + \sigma + c = 1$ .

Let  $\max_Q u = u(x^0, t^0) \geq 0$ ,  $(x^0, t^0) \in Q$ . Suppose that  $\mathcal{L}u \leq -\varepsilon < 0$ . Consider the cylinder  $Q_\rho = \{(x, t) \mid t^0 - \frac{\rho^2}{2} < t < t^0, |x| < \rho\}$  and introduce the function

$$u_\delta(x, t) = u(x, t) - u(x^0, t^0) + \delta \left( 1 - \frac{|x - x^0|^2 - 2(t - t^0)}{\rho^2} \right).$$

For sufficiently small  $\rho$  we have  $u|_{\partial Q_\rho} \leq u(x^0, t^0)$  and therefore  $u_\delta|_{\partial Q_\rho} \leq 0$ . Thus, we can apply Theorem 4.1 with  $p_0 = p_1$ ,  $q_0 = q_1$ . Since  $\mathcal{L}u_\delta \leq -\varepsilon + \frac{\delta}{\rho^2}(2 + 2\rho|b|)$ , this gives for  $\delta < \frac{\varepsilon\rho^2}{4}$

$$\delta = u_\delta(x^0, t^0) \leq M(n, \rho, p_1, \|h\|_{p_1, q_1, (Q_\rho)})^{\frac{n}{p_1}} \cdot \left\| \frac{\left(\frac{2\delta}{\rho}|b| - \frac{\varepsilon}{2}\right)_+}{\sigma^{\frac{1}{q_1}}(\det(a))^{\frac{1}{p_1}} c^{1 - \frac{n}{p_1} - \frac{1}{q_1}}} \right\|_{p_1, q_1, (Q_\rho)}.$$

Since  $\sigma^{\frac{1}{q_1}}(\det(a))^{\frac{1}{p_1}} c^{1 - \frac{n}{p_1} - \frac{1}{q_1}} \leq \mathbf{Sp}(a) + \sigma + c = 1$ , we obtain

$$\delta \leq M^{\frac{n}{p_1}} \cdot \frac{2\delta}{\rho} \cdot \left\| \left(h - \frac{\varepsilon\rho}{4\delta}\right)_+ \right\|_{p_1, q_1, (Q_\rho)} = o(\delta) \quad \text{as } \delta \rightarrow 0.$$

This contradiction proves the statement.  $\square$

## 5 The case of “composite” coefficients

Consider the case where the coefficients  $b_i$  are “composite”, i.e. they can be written in the form (1.2), and

$$\|h_k\|_{p_k, q_k, (Q)} < \infty, \quad h_k = \sigma^{-\frac{1}{q_k}}(\det(a))^{-\frac{1}{p_k}} c^{\frac{n}{p_k} + \frac{1}{q_k} - 1} |b^{(k)}| \quad (5.1)$$

for some  $p_k, q_k$  such that  $\frac{n}{p_k} + \frac{1}{q_k} \leq 1$ . We again suppose that  $c \geq 0$  and other assumptions of Theorem 4.1.

**Theorem 5.1.** *Under mentioned assumptions the estimate (3.5) holds with  $M$  depending on  $n$ ,  $R$ ,  $p_1$  and norms  $\|h_k\|_{p_k, q_k, (Q_u)}$ ,  $k = 1, \dots, m$ .*

*Proof.* We restrict ourselves to the case of smooth coefficients. Further arguments are similar to Section 4.

Let  $p_1 = n$ , and  $p_k > n$  for  $k \geq 2$ . Denote by  $B_k$ ,  $k \geq 2$ , solutions of boundary value problems

$$\mathcal{L}_k B_k = |b^{(k)}| \quad \text{in } C_{RT}; \quad B_k|_{\partial' C_{RT}} = 0.$$

As in part 6 of the proof of Theorem 3.1, we introduce a function

$$f \in \mathcal{C}_0^\infty(B_{R+1}); \quad f(x) > \sup_t (h_1(x, t) \cdot \chi_{\Omega_u(t)}); \quad \|f\|_{n,(\mathbb{R}^n)} \leq 2\|h_1\|_{n,\infty,(Q_u)}.$$

Define  $v$  as the solution of boundary value problem

$$\det(D(Dv)) = \left(\frac{2}{n}\right)^n f^n(1 + |Dv|^2)^{\frac{n}{2}} \quad \text{in } B_{R+1}; \quad v|_{\partial B_{R+1}} = 0$$

and set

$$B_1 := -v; \quad \tilde{B} := \sum_{k \geq 2} B_k; \quad B := B_1 + \tilde{B} \cdot (1 + \|DB_1\|_{C_{RT}}).$$

Then

$$\mathcal{L}B \geq (\det(a))^{\frac{1}{n}} h_1(1 + |DB_1|) - |b| \cdot |DB_1| + \sum_{k \geq 2} |b^{(k)}| \cdot (1 + \|DB_1\|_{C_{RT}}) \geq |b|,$$

and thus  $B$  satisfies the assumptions of Pivotal Lemma.

We apply the estimate (3.1) with  $p = p_k$ ,  $q = q_k$ , with regard to Corollary 3.1, to  $u = \pm B_k$ . Summing over  $k \geq 2$ , we obtain

$$\|\tilde{B}\|_{C_{RT}} \leq N(n) \sum_{k \geq 2} (\|B_1\|_{C_{RT}} + R + (1 + \|DB_1\|_{C_{RT}}) \|\tilde{B}\|_{C_{RT}})^{\frac{n}{p_k}} \cdot \|h_k\|_{p_k, q_k, (C_{RT})}, \quad (5.2)$$

while Lemma 3.1 in [Kr4] gives

$$\|B_1\|_{C_{RT}}, \|DB_1\|_{C_{RT}} \leq N_3(n)(R + 1) \exp(N_4(n)\|h_1\|_{n,\infty,(Q_u)}).$$

Since  $p_k > n$  for  $k \geq 2$ , (5.2) easily implies the statement of Theorem.  $\square$

**Remark 5.1.** The proof scheme of Theorem 5.1 is taken from [Kr4], where such proof was given in some particular cases.

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